Recap: Part I

Autonomous systems:

$$\dot{x} = f(x)$$

- check for eqlb. points $\quad f(x) = 0$
- Linearize $\quad A = \frac{\partial f}{\partial x}$

  if A is Hurwitz $\Leftrightarrow$ exp stable.
- Find a Lyapunov function, $V(x) > 0 \ \forall x \neq 0$

  $V(0) = 0$

and show

$\frac{\partial V}{\partial x} f(x)$ $\quad$ set containing 0

- $\dot{V}(x) \leq 0 \quad \forall x \in D \quad \Rightarrow$ stable
- $\dot{V}(x) < 0 \quad \forall x \in D - \{0\} \Rightarrow$ AS
- $\dot{V}(x) < 0 \quad \forall x \neq 0 \quad \Rightarrow$ GAS

  $+$ V is radially unbounded.

- LaSalle's invariance principle:

$x(t) \to$ largest invariant set in $E = \{x \mid \dot{V}(x) = 0\}$

- if $x(t) = 0$ is the only solution that stays in E

$\Rightarrow$ AS

- Globally exp. stable

$$c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2$$

$$\dot{V}(x) \leq -c_3\|x\|^2$$

$\Rightarrow$ globally exp. stable

- Converse is also true

if globally exp. stable $\Rightarrow \exists\, V$ s.t.

...

- Used to prove stability under vanishing perturbations.

Input-output systems:

$$u \longrightarrow \boxed{H} \longrightarrow Y$$

- L-stability:

increasing & $\alpha(0) = 0$

$$\|Y\|_L \leq \alpha(\|u\|_L) + \beta$$

with finite-gain:

$$\|Y\|_L \leq \gamma \|u\|_L + \beta$$

- Lyapunov function to show L-stability

Thm. 5.1:

$$\dot{x} = f(x,u), \quad \|f(x,u) - f(x,0)\| \leq L\|u\|$$
$$Y = h(x,u) \quad \|h(x,u)\| \leq \eta_1\|x\| + \eta_2\|u\|$$

$\exists\ V$ s.t.

$$c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2$$

$$\frac{\partial V}{\partial x} f(x,0) \leq -c_3\|x\|^2$$

$$\left\|\frac{\partial V}{\partial x}\right\| \leq c_4\|x\|$$

$\Longrightarrow$ Lp-stable with finite-gain

Computing the L_2-gain:

- Linear sys. $\sup_\omega \|G(j\omega)\|_2$

  thm. 5.4

- Control-affine: $\dot{x} = f(x) + g(x)u$

  thm. 5.5 $y = h(x)$

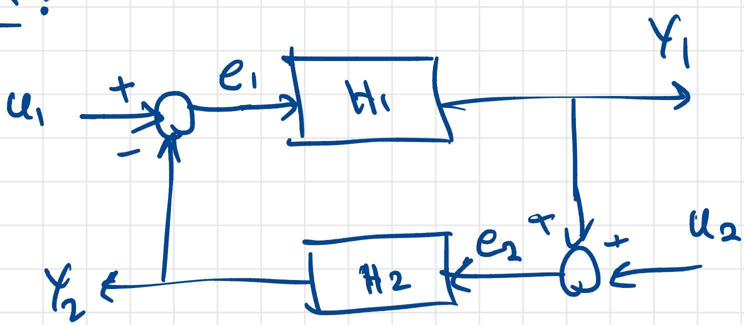
HJ ineq.

$$\frac{\partial V}{\partial x} f + \frac{1}{2\gamma^2} \frac{\partial V}{\partial x} g g^T \frac{\partial V}{\partial x}^T + \frac{1}{2}\|h\|^2 \leq 0$$

$\Rightarrow$ gain $\leq \gamma$

Small-gain thm:

thm. 5.6



$$\|y_1\|_L \leq \gamma_1 \|e_1\|_L + \beta_1$$

$$\|y_2\|_L \leq \gamma_2 \|e_2\|_L + \beta_2$$

if $\gamma_1 \gamma_2 < 1 \Rightarrow$ FB is L-stable

Passivity:

$$\dot{x} = f(x,u)$$
$$y = h(x,u)$$

- if $\exists$ storage function V s.t. $V(x) \geq 0$

$\frac{\partial V}{\partial x} f(x,u)$ ← $\dot{V} \leq u^T y \implies$ passive.

$\dot{V} \leq u^T y - y^T \varphi(y) \Rightarrow$ output strictly passive

$\dot{V} \leq u^T y - u^T \varphi(u) \Rightarrow$ input strictly passive

$u^T \varphi(u) > 0$ $\forall x \neq 0$

$\dot{V} \leq u^T y - W(x) \Rightarrow$ strictly passive

$W(x) > 0$ $\forall x \neq 0$

- For memoryless sys. $y = h(u)$, same def but $V = 0$ (Def. 6.1)

- Passive + p.d $V \implies$ stable

Lemma 6.6

- (A) strictly passive $\implies$ AS

Lemma 6.7

- (B) output strictly passive + zero-state obs $\implies$ AS ← uses LaSalle.

Lemma 6.5: Output strictly passive with

$$\dot{V} \leq u^T y - \delta \|y\|^2$$

$\Rightarrow$ L_2-stable with finite gain $\leq \frac{1}{\delta}$

Another way to compute L_2-gain different than HJ

- For linear sys, we look at the transfer function. $G(s)$

Lemma 6.4:

if (strictly) positive real $\Rightarrow$ (strictly) passive.

- Positive real:

① poles are in LHP

② $Re(G(j\omega)) > 0$

③ if $j\omega$ is a pole, then $\lim_{s \to j\omega} (s - j\omega) G(s) \geq 0$

strictly pos. real: $G(s-\varepsilon)$ is pos. real for some $\varepsilon > 0$

Another way to check strict pos. real.

Lemma 6.1: strict pos. real iff

① poles have neg. real part (Hurwitz)

② $Re(G(j\omega)) > 0 \quad \forall \omega \in \mathbb{R}$

③ $\lim_{\omega \to \infty} \omega^2 Re(G(j\omega)) > 0$ or $G(\infty) > 0$

- Feedback connection

Thm 6.1: H_1 and H_2 are passive $\Longrightarrow$ FB is passive

Thm 6.3: if H_1 is (A) or (B)
and H_2 is (A) or (B) $\Longrightarrow$ closed-loop stable

Thm. 6.2: (For L_2-Stability)

- if
$$\dot{V}_1 \leq e_1^T y_1 - \varepsilon_1 \|e_1\|^2 - \delta_1 \|y_1\|^2$$
$$\dot{V}_2 \leq e_2^T y_2 - \varepsilon_2 \|e_2\|^2 - \delta_2 \|y_2\|^2$$

if $\varepsilon_1 + \delta_2 > 0$ and $\varepsilon_2 + \delta_1 > 0$. Then

FB sys is output strictly passive and finite-gain L_2-stable

Examples:

1. Consider memoryless sys. $Y = h(u)$ where $h: \mathbb{R}^m \to \mathbb{R}^m$ is globally Lip. with Const. L. Investigate L_p-stability for each $p \in [1, \infty]$ when

a) $h(0) = 0$

b) $h(0) \neq 0$

Solution:

a) h is Lip. $\Rightarrow \|h(u) - h(0)\| \leq L\|u - 0\|$

$$\Rightarrow \underbrace{\|h(u)\|}_{Y} \leq L\|u\|$$

$p = \infty \rightarrow \|Y\|_{L_\infty} = \sup_t \|Y(t)\| \leq L \sup_t \|u(t)\|$

$= L \|u\|_{L_\infty} \Rightarrow L_\infty$-Stable

$p \in [1, \infty) \Rightarrow \|Y\|_{L_p}^p = \int \|Y(t)\|^p \, dt$

$\leq \int L^p \|u(t)\|^p \, dt$

$= L^p \|u\|_{L_p}^p \Rightarrow \|Y\|_{L_p} \leq L \|u\|_{L_p}$

$\Downarrow$

L_p-Stable

b) $h(0) \neq 0 \implies \|h(u) - h(0)\| \leq L\|u\|$

$$\implies \|h(u)\| \leq L\|u\| + \|h(0)\|$$

$p = \infty \to \|y\|_{L_\infty} = \sup_t \|y(t)\|$

$$\leq \sup_t \{ L\|u(t)\| + \|h(0)\| \}$$

$$= L\|u\|_{L_\infty} + \underbrace{\|h(0)\|}_{\text{bias}}$$

$$\implies L_\infty\text{-stable}$$

$p \in [1, \infty) \implies \|y\|_{L_p}^p = \int_0^\infty \|y(t)\|^p \, dt$

$$\leq \int_0^\infty (L\|u(t)\| + \|h(0)\|)^p \, dt \to \text{seems to be unbounded even if } u=0$$

Consider $u(t) = 0 \implies y(t) = h(0)$

$$\implies \|y\|_{L_p}^p = \int_0^\infty \|h(0)\|^p \, dt$$

$$= \infty$$

$\implies$ Not L_p stable for $p \in [1, \infty)$

② Consider

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - x_2 - a\tanh(x_1) + u$$
$$y = x_1$$

where $a \geq 0$. Show Lp-stability with finite gain.

Solution: we use thm. 5.1 $\rightarrow$ find Lyapunov func.

$$V(x) = \frac{x_1^2}{2} + a\int_0^{x_1} \tanh(z)\,dz + \frac{x_2^2}{2}$$

$$\Rightarrow \dot{V}(x) = (x_1 + a\tanh(x_1))\,x_2 + x_2(-x_1 - x_2 - a\tanh(x_1))$$

$$= -x_2^2 \Rightarrow \text{Not enough for } \dot{V} \leq -c\|x\|^2$$

Consider $V(x) = \underbrace{\frac{1}{2}P_{11}x_1^2 + P_{12}x_1x_2 + \frac{1}{2}P_{22}x_2^2}_{\frac{1}{2}x^TPx} + a\int_0^{x_1}\tanh(z)\,dz$

$P_{11} > 0$, $P_{11}P_{22} - P_{12}^2 > 0$

$$\Longrightarrow \dot V(x) = (P_{11} x_1 + a\tanh(x_1) + P_{12} x_2)\, x_2$$
$$+ (P_{22} x_2 + P_{12} x_1)(-x_1 - x_2 - a\tanh(x_1))$$

$$= (P_{11} \cancel{- P_{22} - P_{12})}\, x_1 x_2$$
$$\cancel{a\tanh(x_1)}\, x_2 - P_{22}\, \cancel{a\tanh(x_1)}\, x_2$$
$$- P_{12}\, a\, x_1 \tanh(x_1)$$
$$+ (P_{12} - P_{22})\, x_2^2 - P_{12}\, x_1^2$$

$\longrightarrow$ $P_{22} = 1$ to cancel $a\tanh(x_1) x_2$

we want

$$\left.\begin{array}{l} P_{11} - P_{22} - P_{12} = 0 \\ P_{12} > 0 \\ P_{12} - P_{22} > 0 \\ \text{also} \quad P_{11} P_{22} - P_{12}^2 > 0 \end{array}\right\} \begin{array}{l} P_{22} = 1 \\ P_{12} = 2 \\ P_{11} = 5 \end{array}$$

$$\Longrightarrow \dot V = -2a\, x_1 \tanh(x_1) - x_2^2 - 2x_1^2$$
$$\le -\|x\|^2 \longrightarrow c_3 = 1$$

$$V(x) = \frac{1}{2} x^T P x + a\int_0^{x_1} \tanh(z)\, dz$$

To show $c_1\|x\|^2 \le V(x) \le c_2\|x\|^2$, we use

$$\lambda_{min}(P)\|x\|^2 \le x^T P x \le \lambda_{max}(P)\|x\|^2$$

and $$0 \le \int_0^{x_1} \tanh(z)\,dz \le \int_0^{x_1} z\,dz \le \frac{x_1^2}{2} \le \frac{\|x\|^2}{2}$$

$\tanh(z) \le z$

$$\Longrightarrow \underbrace{\frac{1}{2}\lambda_{min}(P)}_{c_1}\|x\|^2 \le V(x) \le \underbrace{\left(\frac{1}{2}\lambda_{max}(P)+\frac{1}{2}\right)}_{c_2}\|x\|^2$$

- It is also clear that

$$|f(x,u) - f(x,0)| \le |u|$$

and $$|h(x,u)| \le \|x\|$$

→ Apply Thm 5.1

See Example 5.5 for different Lyapunov func

③ Consider

See Examples 5.7, 5.10
6.5

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

Suppose $\exists$ p.d function $W(x)$ s.t.

$$\frac{\partial W}{\partial x} f(x) \leq 0 \qquad \forall x$$

$$\frac{\partial W}{\partial x} g(x) = h^T(x)$$

Show that the system is

a) passive

b) assume zero-state obs. Then, show AS with output feedback $u = -Ky$ where $K > 0$

c) Let $u = -Ky + v$. Show that the sys is output strictly passive. and finite-gain L_2-stable.

Solution:

a) Take $V(x) = W(x)$ as storage func.

Then,
$$\dot{V} = \frac{\partial V}{\partial x} f(x,u)$$
$$= \frac{\partial W}{\partial x}(f(x) + g(x)u)$$
$$\leq \frac{\partial W}{\partial x} g(x) u = h(x)^T u$$
$$= y^T u \Rightarrow \text{passive!}$$

b) let $u = -ky$

$$\left.\begin{array}{l} \Rightarrow \quad \dot{V} \leq -k\|y\|^2 \\ \text{and} \quad \text{Zero-State obs} \end{array}\right\} \Rightarrow \text{A.S.}$$

c) $u = -ky + v \Rightarrow \dot{V} \leq -k\|y\|^2 + y^T v \Leftarrow$ output strictly passive.

To show finite L_2-gain, we can follow two approaches.

Approach 1:

$$\dot{V} \leq -K\|y\|^2 + y^T v \leq -K\|y\|^2 + \frac{K}{2}\|y\|^2 - \frac{1}{2K}\|v\|^2$$

$$\overset{\text{integrate}}{\Longrightarrow} \quad V(x(t)) - V(x(0)) \leq -\frac{K}{2}\int_0^t \|y(s)\|^2 ds + \frac{1}{2K}\int_0^t \|v(s)\|^2 ds$$

$$\overset{V(x(t))>0}{\Longrightarrow} \quad \frac{K}{2}\int_0^t \|y(s)\|^2 ds \leq \frac{1}{2K}\int_0^t \|v(s)\|^2 ds + V(x(0))$$

$$\overset{t\to\infty}{\Longrightarrow} \quad \|y\|_{L_2}^2 \leq \frac{1}{K^2}\|v\|_{L_2}^2 + \frac{2}{K}V(x(0))$$

$\sqrt{a^2+b^2} \leq a+b$

$$\Longrightarrow \quad \|y\|_{L_2} \leq \frac{1}{K}\|v\|_{L_2} + \sqrt{\frac{2}{K}V(x(0))}$$

$\Longleftarrow$ finite-gain L_2-stable, gain $\leq \frac{1}{K}$

see Lemma 6.5

Approach 2:

$$\dot{x} = f(x) + g(x)(-Kh(x) + v)$$
$$= f(x) - Kg(x)h(x) + g(x)v$$

Using HJ inequality.

- Let $V(x) = \alpha W(x)$, Then, the LHS of HJ ineq.

$$\frac{\partial V}{\partial x}(f(x) - Kg(x)h(x)) + \frac{1}{2\gamma^2}\frac{\partial V}{\partial x} g(x) g(x)^T \frac{\partial V}{\partial x}^T + \frac{1}{2}\|h(x)\|^2 =$$

$$= \alpha \frac{\partial W}{\partial x} f(x) - \alpha K \frac{\partial W}{\partial x} g(x) g(x)^T \frac{\partial W}{\partial x}^T + \frac{\alpha^2}{2\gamma^2}\frac{\partial W}{\partial x} g(x) g(x)^T \frac{\partial W}{\partial x}$$

$$+ \frac{1}{2}\left\|\frac{\partial W}{\partial x} g(x)\right\|^2 \leq \left(\frac{\alpha^2}{2\gamma^2} - \alpha K + \frac{1}{2}\right)\left\|\frac{\partial W}{\partial x} g(x)\right\|^2$$

In order to satisfy HJ ineq, we must have

$$\frac{\alpha^2}{2\gamma^2} - \alpha K + \frac{1}{2} \leq 0, \quad \text{minimize over } \alpha$$

$$\Rightarrow \frac{\alpha}{\gamma^2} = K \Rightarrow \alpha = K\gamma^2 \Rightarrow \frac{-K^2\gamma^2}{2} + \frac{1}{2} \leq 0$$

$$\Rightarrow \gamma \geq \frac{1}{K} \longrightarrow \text{Finite-gain with gain} \geq \frac{1}{K}$$

④ Consider the FB sys with

$$H_1: \begin{cases} \dot{x}_1 = -x_1 + x_2 \\ \dot{x}_2 = -x_1^3 - x_2 + e_1 \\ y_1 = x_2 \end{cases}$$

$$H_2: \begin{cases} \dot{x}_3 = -x_3^3 + e_2 \\ y_2 = \frac{1}{2} x_3^3 \end{cases}$$

show the the FB is L_2-stable with finite gain!

Solution:

① use HJ ineq to compute L_2-gain and small gain thm

② use passivity and thm. 6.2

Let's use passivity

$$V_1(x) = \frac{x_1^4}{4} + \frac{x_2^2}{2}$$

$$\Rightarrow \dot{V}_1 = x_1^3(-x_1 + x_2) + x_2(-x_1^3 - x_2 + e_1)$$

$$= -x_1^4 - x_2^2 + e_1 x_2$$

$$\leq -y_1^2 + y_1 e_1 \longrightarrow \varepsilon_1 = 0, \ \delta_1 = 1$$

$$V_2(x) = \frac{a x_3^4}{4} \Rightarrow \dot{V}_2 = a x_3^3(-x_3^3 + e_2)$$

$a > 0$

$$= -a x_3^6 + a x_3^3 e_2$$

$$\text{let } a = \frac{1}{2} \Rightarrow \dot{V}_2 \cancel{\leq} -\frac{1}{2} x_3^6 + \frac{1}{2} x_3^3 e_2$$

$$= -2 y_2^2 + y_2 e_2$$

$$\hookrightarrow \varepsilon_2 = 0, \ \delta_2 = 2$$

$$\xRightarrow[\text{or lemma 6.8}]{\text{thm 6.2}} \mathcal{L}_2\text{-stable with finite gain} \leq \frac{1}{\min(\delta_1, \delta_2)} = 1$$

⑤ Consider FB sys. Example 6.7.

$$H_1: \begin{cases} \dot{x} = f(x) + g(x) e_1 \\ y_1 = h(x) \end{cases} \qquad H_2: \ y_2 = k e_2$$

where $k > 0$, $e_1, e_2, y_1, y_2 \in \mathbb{R}^p$

Suppose $\exists$ p.d. function V_1 s.t.

$$\frac{\partial V_1}{\partial x} f(x) \leq 0, \quad \frac{\partial V_1}{\partial x} g(x) = h^T(x)$$

show the connection is L_2-stable.

Solution:

- Both components are passive

for H_1:
$$\dot{V}_1 = \frac{\partial V_1}{\partial x} f(x) + \frac{\partial V_1}{\partial x} g(x) e_1$$

$$\leq h(x)^T e_1 = y_1^T e_1 \quad \rightarrow \varepsilon_1 = \delta_1 = 0$$

for H_2:
$$e_2^T y_2 = k e_2^T e_2 = \alpha k e_2^T e_2 + (1-\alpha) \frac{1}{k} y_2^T y_2$$

$$\Leftrightarrow e_2^T y_2 = \alpha k \|e_2\|^2 + \frac{1-\alpha}{k} \|y_2\|^2$$

for memoryless sys, $V_2 = 0$

$$\Rightarrow \quad 0 = \dot{V}_2 \leq e_2^T Y_2 - \alpha K \|e_2\|^2 - \frac{1-\alpha}{K}\|Y_2\|^2$$

With $\alpha \in (0,1)$ $\rightarrow$ $\varepsilon_2 = \alpha K > 0$

$\delta_2 = \frac{1-\alpha}{K} > 0$

$\Rightarrow$ Apply thm 6.2 to conclude L_2-stability with finite gain.